



RBE 2004

ระบบอัตโนมัติ (Automatic System)

สาขาวิศวกรรมหุ่นยนต์

คณะวิศวกรรมศาสตร์และเทคโนโลยีอุตสาหกรรม

มหาวิทยาลัยราชภัฏสวนสุนันทา

Chapter 5 The Stability of Linear Feedback Systems

Lecture 7

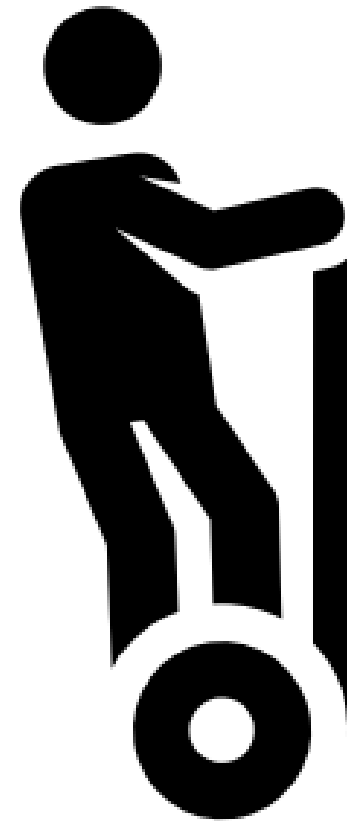
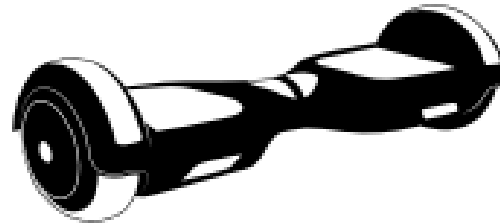
- Concept of Stability
- Routh-Hurwitz Stability Criterion
- Root Locus Method
- Computer Simulation (Matlab/Simulink)

Objective

- Understand the principle of stability
- Define the conditions required for stability
- Apply to the Routh – Hurwitz stability criterion

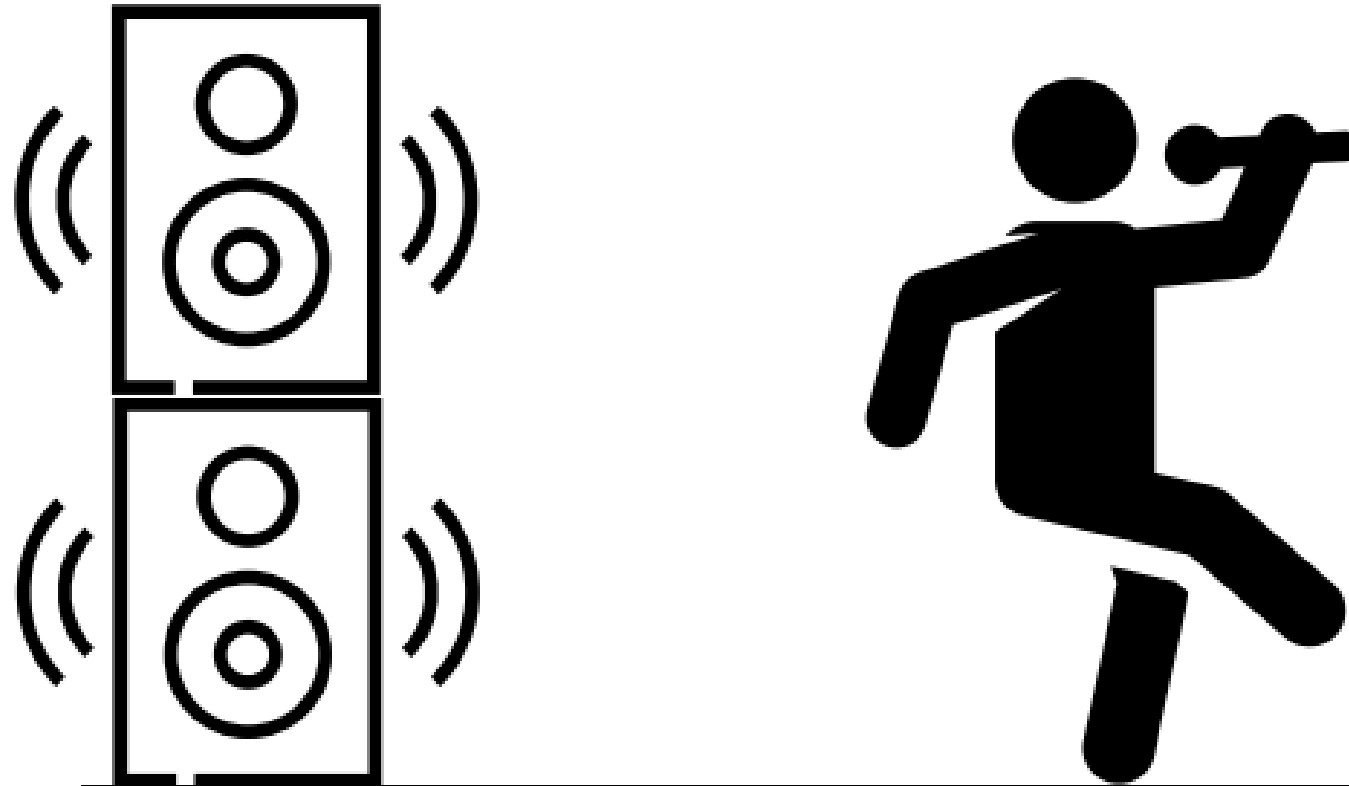
Applications

Are these control systems stable without feedback control?



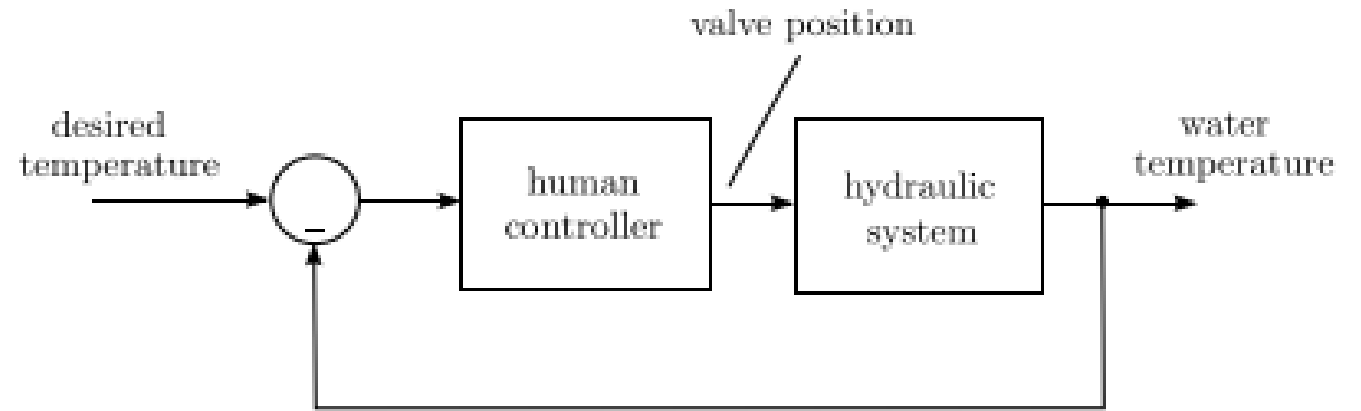
Applications

Example of destabilizing positive feedback.



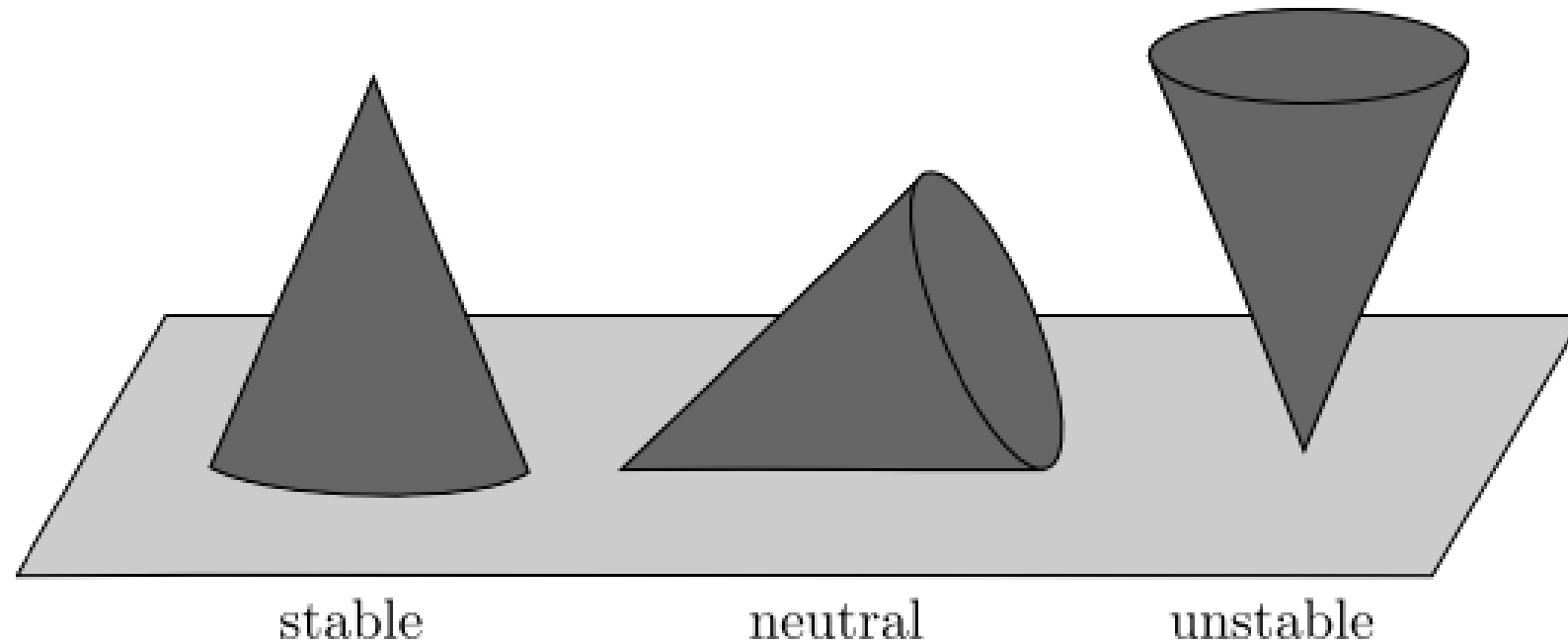
Applications

High control loop gains can make a system unstable.



The concept of stability

A stable system is a dynamic system with a bounded response to a bounded input.



A system is stable if all closed-loop transfer function poles lie in the left-half s -plane.

Requirements for stability

Consider the generic transfer function:

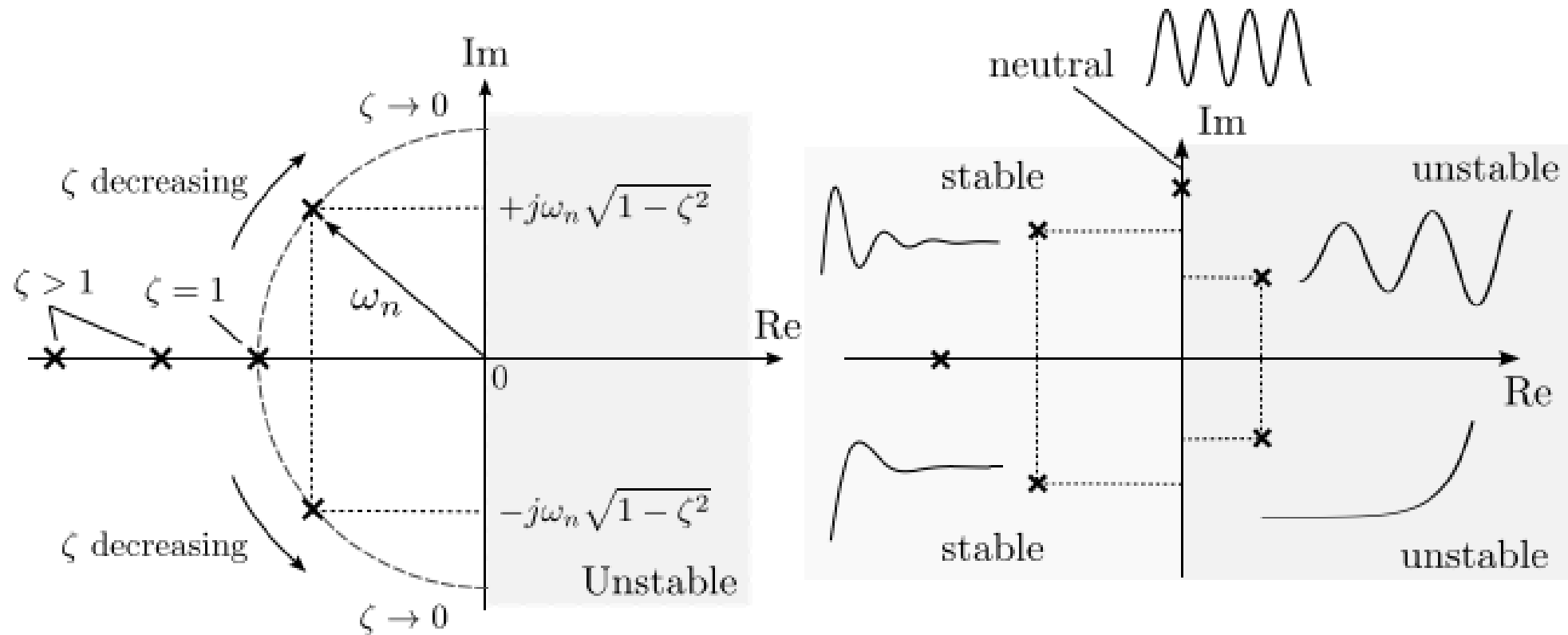
$$T(s) = \frac{p(s)}{q(s)} = \frac{k \prod_{i=1}^M (s + z_i)}{s^N \prod_{k=1}^Q (s + \sigma_k) \prod_{m=1}^R [s^2 + 2\alpha_m s + (\alpha_m^2 + \omega_m^2)]} \quad (1)$$

The output response for an impulse function input and $N = 0$ is

$$y(t) = \sum_{k=1}^Q A_k e^{-\sigma_k t} + \sum_{m=1}^R B_m \left(\frac{1}{\omega_m} \right) e^{-\alpha_m t} \sin(\omega_m t + \theta_m) \quad (2)$$

A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have **negative real parts**

Stability and the location of poles



Stable system: Poles are in the left-half plane

Neutral system or marginally stable: Poles are purely imaginary ($j\omega$)

Unstable system: At least one of the poles is in the right-half plane

Marginally stable

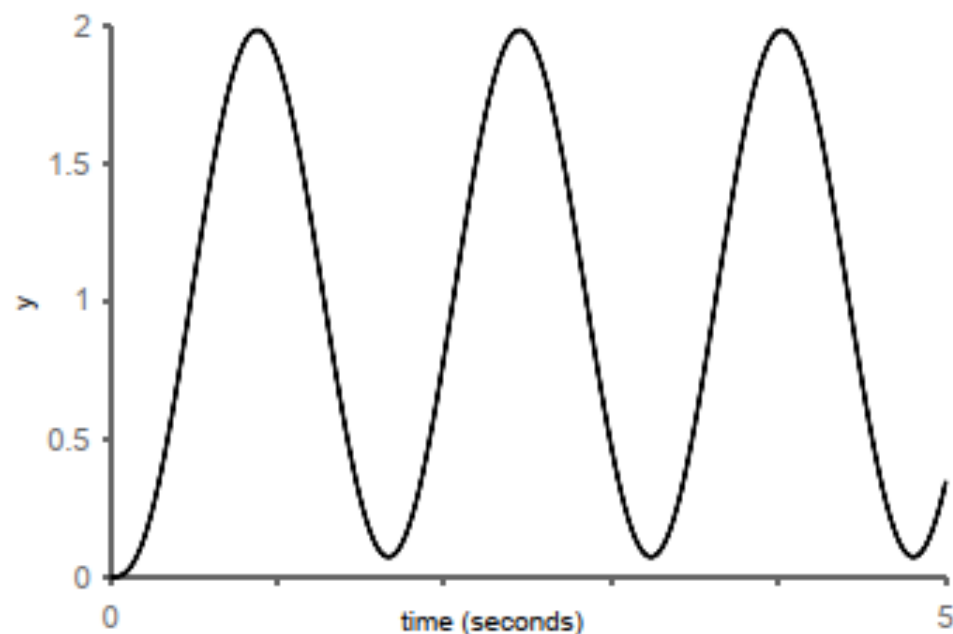
Some poles of the closed-loop transfer functions are purely imaginary.

Example: The transfer function

$$T(s) = \frac{1}{(s + 10)(s^2 + 16)} \quad (3)$$

has the poles $s_1 = -10$, $s_2 = 4j$, $s_3 = -4j$.

For $r(t) = 1$:



The Routh-Hurwitz criterion

This criterion is a necessary and sufficient condition for stability

Order the coefficient of the characteristic equation

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{s-1} + \dots + a_1 s + a_0 = 0 \quad (4)$$

Into an array as follows:

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \quad (5)$$

The number of roots with positive real parts is equal to the number of changes in sign of the first column.

The Routh-Hurwitz criterion

Step 1: Place the highest order of $q(s)$ on the top-left column from n to 0.

s^n	a_n	a_{n-2}	a_{n-4}	\dots
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	\dots
s^{n-2}	b_{n-1}	b_{n-3}	b_{n-5}	\dots
s^{n-3}	c_{n-1}	c_{n-3}	c_{n-5}	\dots
\vdots	\vdots	\vdots	\vdots	
s_0	h_{n-1}			

(6)

Step 2: From the second column, the first two rows are the coefficients of

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{s-1} + \dots + a_1 s + a_0 = 0$$

s^n	a_n	a_{n-2}	a_{n-4}	\dots
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	\dots
s^{n-2}	b_{n-1}	b_{n-3}	b_{n-5}	\dots
s^{n-3}	c_{n-1}	c_{n-3}	c_{n-5}	\dots
\vdots	\vdots	\vdots	\vdots	
s_0	h_{n-1}			

(7)

The Routh-Hurwitz criterion

Step 3: Fill out the reminder rows

$$\begin{array}{cccc} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \quad (8)$$

For the b_n coefficients:

$$b_{n-1} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} \quad (9)$$

$$b_{n-3} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} \quad (10)$$

The Routh-Hurwitz criterion

Step 3: Fill out the remainder rows

$$\begin{array}{cccc} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ s^{n-3} & \boxed{c_{n-1}} & \boxed{c_{n-3}} & \boxed{c_{n-5}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \quad (11)$$

For the c_n coefficients:

$$c_{n-1} = \frac{-1}{b_{n-1}} \left\| \begin{array}{cc} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{array} \right\| \quad (12)$$

$$c_{n-3} = \frac{-1}{b_{n-1}} \left\| \begin{array}{cc} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{array} \right\| \quad (13)$$

And so on...

Example of Routh-Hurwitz matrix

$$q(s) = s^5 + 2s^4 + 1s^3 + 4s^2 + 11s + 10$$

The Routh-Hurwitz matrix is

$$\begin{array}{c|ccc} s^5 & 1 & 1 & 11 \\ s^4 & 2 & 4 & 10 \\ \hline s^3 & & & \\ \hline s^2 & & & \\ \hline s^1 & & & \\ \hline s^0 & & & \end{array}$$

$$b_{n-1} = \frac{-1}{a_{n-1}} \left\| \begin{array}{cc} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{array} \right\|$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \left\| \begin{array}{cc} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{array} \right\|$$

Example of Routh-Hurwitz matrix

$$q(s) = 3s^5 + s^4 + 2s^3 + 1s^2 + 1$$

The Routh-Hurwitz matrix is

$$\begin{array}{c|ccc} s^5 & 3 & 2 & 0 \\ s^4 & 1 & 1 & 1 \\ \hline s^3 & & & \\ \hline s^2 & & & \\ \hline s^1 & & & \\ \hline s^0 & & & \end{array}$$

$$b_{n-1} = \frac{-1}{a_{n-1}} \left\| \begin{array}{cc} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{array} \right\|$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \left\| \begin{array}{cc} a_{n-1} & a_{n-3} \\ b_{n-1} & a_{n-3} \end{array} \right\|$$

The Routh-Hurwitz Criterion

Stability requires all roots of $q(s)$ to have positive real parts.

→ **Count the sign *changes* in the first column**

→ **That is the number of roots in the right half plane**

Case 1: All elements in the first column are nonzero

$$q(s) = a_2s^2 + a_1s + a_0$$

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & a_0 & 0 \end{array}$$

Thus the system is stable if:

→ $a_2 > 0$, **and** $a_1 > 0$, **and** $a_0 > 0$.

or

→ $a_2 < 0$, **and** $a_1 < 0$, **and** $a_0 < 0$.

The Routh-Hurwitz Criterion

Case 2: There is a zero in the first column. Other elements in the row containing the zero are nonzero.

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Replace the zero in the first column with $\epsilon \rightarrow 0^+$, i.e. $0 < \epsilon \ll 1$

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & \epsilon & 6 & 0 \\ s^2 & c_1 & 10 & 0 \\ s_1 & d_1 & 0 & 0 \\ s_0 & 10 & 0 & 0 \end{array}$$

where:

$$c_1 = \frac{4\epsilon - 12}{\epsilon}, \quad d_1 = \frac{6c_1 - 10\epsilon}{c_1}$$

Now, make $\epsilon \rightarrow 0^+$ and evaluate the first column elements signs

The Routh-Hurwitz Criterion

Case 3: There is a zero in the first column and the other elements of the row containing the zero are also zero.

Case 4: Repeated roots of the characteristic equation on the imaginary axis.

What does it mean?

⇒ The characteristic equations has purely imaginary roots

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & k \\ s_1 & \frac{8-k}{2} & \\ s_0 & k & 0 \end{array}$$

Exercise 45

Find the Routh-Hurwitz matrix for the following closed-loop transfer functions and assess the stability of each system.

$$T(s) = \frac{1}{s^2 + 2s + 1}$$

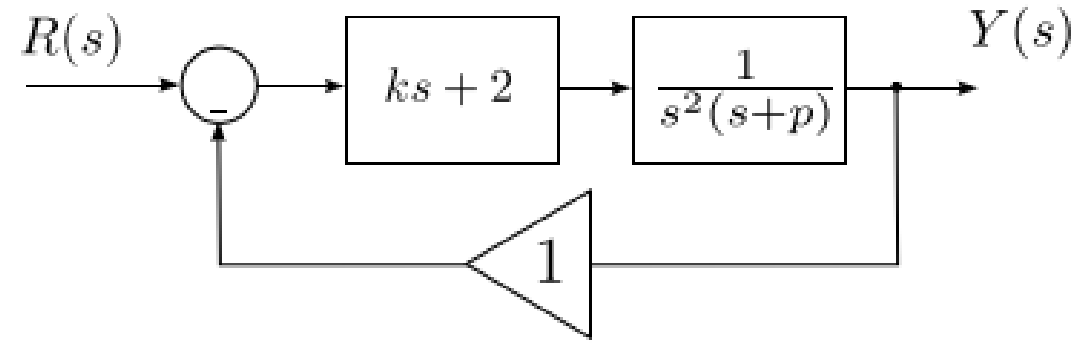
$$R(s) = \frac{1 + s}{3s^3 + s^2 + 2s + 3}$$

$$P(s) = \frac{1}{s^4 + 2s^3 - 100s - 500}$$

$$H(s) = \frac{s^2 + 1}{s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63}$$

Exercise 46

A closed-loop feedback system is shown in the figure.



For what range of values of the parameters k and p is the system stable ?

Procedure:

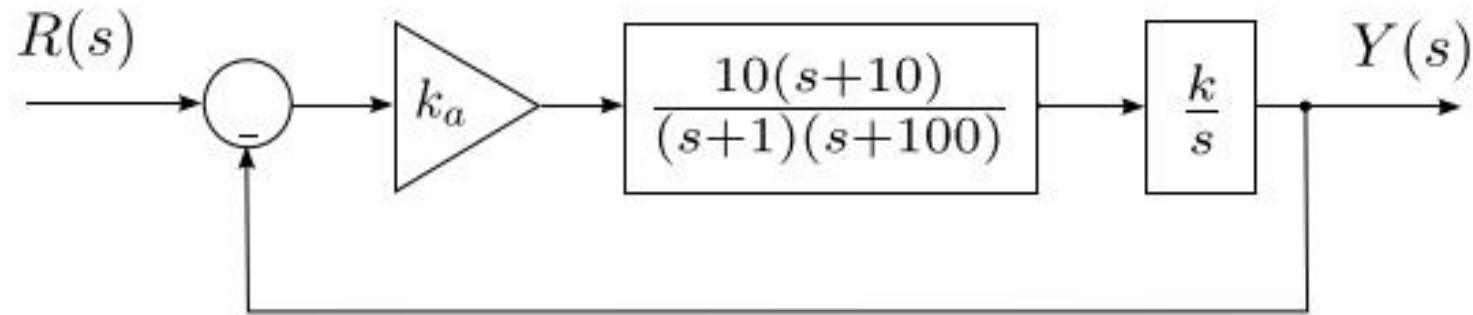
→ Find the closed-loop transfer function

→ Write the Routh-Hurwitz matrix

→ Determine the values of p and k that meet the stability condition

Exercise 47

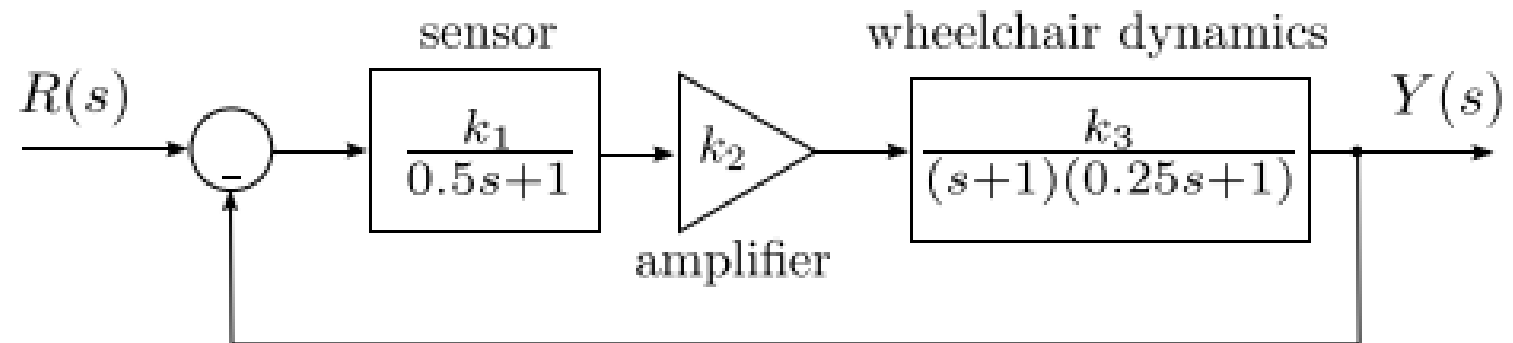
The linear model of a phase detector can be represented by the diagram shown. It is designed to maintain a zero phase between the input carrier signal and local voltage controller oscillator. We want to minimize the steady-state error for a ramp input.



- Determine the maximum gain $k_a k$ in order to maintain a stable system.
- Find $k k_a$ for a steady-state error of 1° for a ramp signal of 100 rad/s .

Exercise 48

A wheelchair velocity control system is shown in the diagram.



Determine the maximum gain $k_1k_2k_3$ for a stable system.

Exercise 49

A teleoperated control system incorporates both a person and a remote machine. In the case of remote operation of a robot, force feedback is useful. The characteristic equation of such a system is

$$s^4 + 20s^3 + k_1s^2 + 4s + k_2 = 0$$



where k_i is a feedback force amplification factor. Determine and plot the region of stability for this system for k_1 and k_2 .

Routh-Hurwitz Stability Criterion